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# Joule-Thomson coefficient of ideal anyons within fractional exclusion statistics

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The analytical expressions of the Joule-Thomson coefficient for homogeneous and harmonically trapped three-dimensional ideal anyons which obey Haldane fractional exclusion statistics are derived. For an ideal Fermi gas, the Joule-Thomson coefficient is negative, which means that there is no maximum Joule-Thomson inversion temperature. With careful study, it is found that there exists a Joule-Thomson inversion temperature in the fractional exclusion statistics model. Furthermore, the relations between the Joule-Thomson inversion temperature and the statistical parameter $g$ are investigated.

*Keywords*: Joule-Thomson coefficient; fractional exclusion statistics; Joule-Thomson inversion temperature

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## I. INTRODUCTION

The fractional exclusion statistics of Haldane is an intermediate statistics between Bose-Einstein and Fermi-Dirac statistics. It is also a generalized dimensional-independent statistics based on state counting methods and is suitable to describe interacting many-particle systems in condensed matter [1–3]. By considering $N$ particles in a $d$-dimensional Hilbert space at fixed size and boundary conditions, the linear relation between the changes of the single-particle space $\Delta d$ and the changes of the particle number $\Delta N$ is defined as $\Delta d = -g\Delta N$ with a parameter $g$ given by Haldane [1].

In Haldane fractional exclusion statistics, the number of microscopic quantum states $W$ of $N$ identical particles occupying a group of $G$ states is

$$W = \prod_i \frac{[G_i + (N_i - 1)(1 - g)]!}{N_i![G_i - gN_i - (1 - g)]!}, \qquad (1)$$

which is interpolated by Johnson, Canright and Wu [2, 3]. It corresponds to the result of Bose-Einstein statistics when the weight factor $g = 0$ and to the one of Fermi-Dirac statistics when $g = 1$. Here the statistical parameter $g$ is described as the change in the number of available states when one particle is added to the system.

The anyon statistical model has been extensively studied in the literature [4–11]. The thermodynamic solution of the one-dimensional ideal anyon gas which obeys the Haldane statistics is equivalent to the Bethe ansatz solution of the Calogero-Sutherland model [12–15]. Furthermore, the thermodynamic extension to $d \geqslant 1$ dimensions of the Calogero-Sutherland model is explored by Potter et al. [4]. The Haldane's fractional exclusion statistics is widely used to describe the low-dimensional condensed matter physics. In the literature, the properties of spinons are characterized in terms of the Haldane's fractional exclusion statistics with corresponding models ( e.g., the Haldane-Shastry model in which free spinons exist [16–18], the one-dimensional supersymmetric t-J model in which the thermodynamics of spinons is investigated [19–21], and the Wess-Zumino-Witten model [22–24]). Besides, the Hubbard model with an infinite-range interaction is also used to study particles obeying Haldane fractional statistics [25, 26].

In thermodynamics, the fact that the temperature changes with the decrease of pressure during an adiabatic or throttling expansion is called the Joule-Thomson effect. The adiabatic expansion or Joule-Thomson process describes the procedure that a gas is forced through a porous plug without heat exchange with the environment. This effect can be described by the Joule-Thomson coefficient $u_{JT}$, which is the partial derivative of temperature with respect to pressure at constant enthalpy

$$u_{JT} \equiv \left(\frac{\partial T}{\partial P}\right)_H, \qquad (2)$$

where $P$ is pressure, $T$ is system temperature and $H$ is enthalpy. If $u_{JT} > 0$, it shows that the temperature of the system decreases during the adiabatic expansion. On the other hand, if $u_{JT} < 0$, the temperature increases in the throttling process. By setting $u_{JT} = 0$, one can calculate the maximum Joule-Thomson inversion temperature. Below this inversion temperature, a free adiabatic expansion causes a decrease in temperature, while it causes a temperature increase above this inversion temperature [27].

There have been many works on the Joule-Thomson coefficient. The Joule-Thomson coefficients of ideal quantum systems are obtained by Ref. [28]. The corresponding results for weakly interacting Fermi and Bose gases are given by means of the pseudopotential method [29]. Recently, the Joule-Thomson coefficient for a $d$-dimensional ideal Bose gas was derived in a power-law potential [30]. The Joule-Thomson coefficient for a strongly

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interacting Fermi gas was also discussed within the quasi-linear approximation framework [31].

According to Landau's phenomenological theory, there is an important parameter called effective mass which is used to characterize the quasiparticle and is determined by the low-temperature isochore heat capacity. If ideal anyons are considered as quasiparticles to describe interactions [32–36], the effective mass is a significant physical quantity in the fractional exclusion statistics model.

In this paper, the analytical expressions of Joule-Thomson coefficients for homogeneous and harmonically trapped three-dimensional ideal anyons are derived within the Haldane fractional exclusion statistics. Furthermore, the Joule-Thomson inversion temperature varying with the statistical parameter $g$ is plotted and the effective mass is calculated. The outline is as follows. In Section II, the distribution function of Haldane fractional exclusion statistics is obtained by the method of Lagrange multiplier. The Joule-Thomson coefficients of homogeneous and harmonically trapped ideal anyons within fractional exclusion statistics are derived analytically in Section III. Effective mass is calculated in Section IV. The numerical calculations and the Joule-Thomson inversion temperature plotted as a function of parameter $g$ are obtained in Section V. A summary is given in the final section.

## II. DISTRIBUTION FUNCTION OF HALDANE FRACTIONAL EXCLUSION STATISTICS

Let us set two Lagrange multipliers $\alpha = -\mu/T$ and $\beta = 1/T$, where $\mu$ is the chemical potential and the natural units $k_B = \hbar = 1$ are used. The average occupation number is defined as $\bar{N}_i \equiv N_i/G_i$. According to the Lagrange multiplier method $\delta \ln W - \alpha\delta N - \beta\delta E = 0$, the most probable distribution function can be derived as [3]

$$\bar{N} = \frac{1}{\omega + g}, \tag{3}$$

where the statistical parameter $g$ and $\omega$ are related as

$$\epsilon = \mu + T\left[(1-g)\ln(1+\omega) + g\ln\omega\right], \tag{4}$$

where $\epsilon$ is the single-particle energy. When $g = 1$, the distribution function $\bar{N}$ goes back to fermionic distribution function, and when $g = 0$ it becomes a bosonic one. According to equation (3), when the temperature is zero, one can get $\bar{N} = 0$ with $\epsilon > \mu$, and $\bar{N} = 1/g$ with $\epsilon < \mu$, which indicates that $g$ characterizes the generalized Pauli principle since the maximum value of the occupation number for a single-particle state is $1/g$.

We define the fugacity $z$ in terms of the chemical potential $\mu$ and introduce a parameter $\omega_0$ as follows:

$$z \equiv exp\left(\frac{\mu}{T}\right) = (1+\omega_0)^{g-1}\omega_0^{-g}. \tag{5}$$

Inserting equation (5) into equation (4), one obtains

$$\epsilon = T\left[(1-g)\ln\left(\frac{1+\omega}{1+\omega_0}\right) + g\ln\left(\frac{\omega}{\omega_0}\right)\right]. \tag{6}$$

## III. THE JOULE-THOMSON COEFFICIENT GIVEN BY THE FRACTIONAL EXCLUSION STATISTICS

### A. Homogeneous gas

The density of states is $D(\epsilon) = (2m)^{3/2}V\epsilon^{1/2}/(2\pi^2)$ with two degrees of the spin degeneracy for a homogeneous gas, where $m$ is the particle mass and $V$ is the system volume.

The expression for the grand thermodynamic potential $\Omega$ is [4–8]

$$\begin{aligned}\Omega &= -PV \\ &= -T\int_0^\infty D(\epsilon)\ln\left(1+\frac{1}{\omega}\right)d\epsilon \\ &= -\frac{TV(2m)^{3/2}}{3\pi^2}\left[\epsilon^{3/2}\ln\left(1+\frac{1}{\omega}\right)\Big|_{\epsilon=0}^{\epsilon=\infty}\right. \\ &\quad\left. -\int_{\omega_0}^\infty \epsilon^{3/2} d\ln\left(1+\frac{1}{\omega}\right)\right].\end{aligned} \tag{7}$$

In the one-dimensional supersymmetric t-J model, three species of particles will contribute to the grand thermodynamic potential by applying an external magnetic field, respectively [20]. In the literature, it is pointed out that the one-dimensional supersymmetric t-J model is equivalent to Haldane fractional statistics with properly fixed statistical parameters [20]. Here, we limit to the three-dimensional free particles in terms of the Haldane's fractional exclusion statistics without magnetic field. The corresponding thermodynamic potential is given by equation (7).

With the help of equation (6), equation (7) can be reduced to

$$P = \frac{2T}{\lambda^3}G_{5/2}(z,g), \tag{8}$$

where the thermal de Broglie wavelength is defined as $\lambda = \sqrt{2\pi/(mT)}$ and

$$G_n(z,g) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}dx}{\omega + g} \tag{9}$$

is the Calogero-Sutherland integral function [4]. $\Gamma(n) \equiv (n-1)! = \int_0^\infty exp(-y)y^{n-1}dy$ is the gamma function and $x = \epsilon/T$. The Calogero-Sutherland integral function has been proved to satisfy the recurrence relation which is the same as the Bose-Einstein and Fermi-Dirac integral functions [4]. The recurrence relation is

$$z\frac{\partial}{\partial z}G_n(z,g) = G_{n-1}(z,g). \tag{10}$$

By turning the variable $\epsilon$ into $\omega$ through equation (6), equation (9) can be reduced to

$$G_n(z,g) = \frac{1}{\Gamma(n)} h_{n-1}(\omega_0, g), \tag{11}$$

where

$$h_n(\omega_0,g) = \int_{\omega_0}^{\infty} \frac{\left\{ \ln\left[ \left(\frac{\omega}{\omega_0}\right)^g \left(\frac{\omega+1}{\omega_0+1}\right)^{1-g} \right] \right\}^n d\omega}{\omega(\omega+1)}. \tag{12}$$

The finite-temperature particle number density and internal energy density can be represented as

$$\begin{aligned} n &= \frac{1}{V}\int_0^{\infty} \frac{D(\epsilon)d\epsilon}{\omega+g} \\ &= \frac{2}{\lambda^3} G_{3/2}(z,g), \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{E}{V} &= \frac{1}{V}\int_0^{\infty} \frac{\epsilon D(\epsilon)d\epsilon}{\omega+g} \\ &= \frac{3T}{\lambda^3} G_{5/2}(z,g) \\ &= \frac{3}{2}P. \end{aligned} \tag{14}$$

At zero temperature, the particle number is $N = (1/g)\int_0^{\widetilde{E}_F} D(\epsilon)d\epsilon = (2mE_F)^{3/2}V/(3\pi^2)$, where $E_F$ is the uniform ideal Fermi energy and $\widetilde{E}_F$ satisfies $\widetilde{E}_F = g^{2/3}E_F$. By replacing the particle number of the ground state into equation (13), one can get

$$\begin{aligned} \frac{3\pi^{1/2}}{4}\left(\frac{T}{T_F}\right)^{3/2} G_{3/2}(z,g) &= \frac{3}{2}\left(\frac{T}{T_F}\right)^{3/2} h_{1/2}(\omega_0,g) \\ &= 1 \end{aligned} \tag{15}$$

with the Fermi characteristic temperature $T_F$ for a uniform ideal Fermi gas.

From equation (13), the partial derivative of particle number density $n$ with respect to temperature $T$ at constant $n$ is written as

$$\begin{aligned} \left(\frac{\partial n}{\partial T}\right)_n &= 2\left(\frac{m}{2\pi}\right)^{3/2} T^{1/2} \left[ \frac{3}{2} G_{3/2}(z,g) \right. \\ &\quad + T\left(\frac{\partial G_{3/2}(z,g)}{\partial T}\right)_{\mu} \\ &\quad \left. + T\left(\frac{\partial G_{3/2}(z,g)}{\partial \mu}\right)_T \left(\frac{\partial \mu}{\partial T}\right)_n \right]. \end{aligned} \tag{16}$$

By combining equation (16) with $(\partial n/\partial T)_n = 0$, the partial derivative of chemical potential to temperature takes the form

$$\left(\frac{\partial \mu}{\partial T}\right)_n = \ln z - \frac{3G_{3/2}(z,g)}{2G_{1/2}(z,g)}. \tag{17}$$

According to the constant total particle number $N$, one can have

$$\begin{aligned} \left(\frac{\partial P}{\partial T}\right)_V &= \left(\frac{\partial P}{\partial T}\right)_{N,V} \\ &= \frac{1}{\lambda^3}\left[ 5G_{5/2}(z,g) - \frac{3G_{3/2}^2(z,g)}{G_{1/2}(z,g)} \right], \end{aligned} \tag{18}$$

$$\begin{aligned} \left(\frac{\partial V}{\partial T}\right)_P &= N\left(\frac{\partial (1/n)}{\partial T}\right)_P \\ &= \frac{N\lambda^3}{4TG_{3/2}(z,g)} \\ &\quad \times \left[ \frac{5G_{5/2}(z,g)G_{1/2}(z,g)}{G_{3/2}^2(z,g)} - 3 \right]. \end{aligned} \tag{19}$$

To derive the expression of the Joule-Thomson coefficient, the isochore heat capacity $C_V$ and isobar heat capacity $C_P$ per particle are derived first as

$$\begin{aligned} \frac{C_V}{N} &= \frac{1}{N}\left(\frac{\partial E}{\partial T}\right)_{N,V} \\ &= \frac{15G_{5/2}(z,g)}{4G_{3/2}(z,g)} - \frac{9G_{3/2}(z,g)}{4G_{1/2}(z,g)} \end{aligned} \tag{20}$$

$$= \frac{5h_{3/2}(\omega_0,g)}{2h_{1/2}(\omega_0,g)} - \frac{9h_{1/2}(\omega_0,g)}{2h_{-1/2}(\omega_0,g)}, \tag{21}$$

$$\begin{aligned} \frac{C_P}{N} &= \frac{C_V}{N} + \frac{T}{N}\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P \\ &= \frac{5h_{3/2}(\omega_0,g)}{6h_{1/2}(\omega_0,g)} \\ &\quad \times \left[ \frac{5h_{3/2}(\omega_0,g)h_{-1/2}(\omega_0,g)}{3h_{1/2}^2(\omega_0,g)} - 3 \right]. \end{aligned} \tag{22}$$

Furthermore, from the fundamental thermodynamic relations and equations (2), (19) and (22), the Joule-Thomson coefficient can be given by

$$\begin{aligned} u_{JT} &= \frac{1}{C_P}\left[ T\left(\frac{\partial V}{\partial T}\right)_P - V \right] \\ &= \frac{\pi^{1/2}\lambda^3}{2}\left[ \frac{1}{2h_{3/2}(\omega_0,g)} \right. \\ &\quad \left. - \frac{h_{-1/2}(\omega_0,g)}{5h_{3/2}(\omega_0,g)h_{-1/2}(\omega_0,g) - 9h_{1/2}^2(\omega_0,g)} \right]. \end{aligned} \tag{23}$$

### B. Harmonically trapped gas

We define the geometric mean of the trap frequencies as $\varpi = (\omega_x\omega_y\omega_z)^{1/3}$. The corresponding density of states is $D(\epsilon) = \epsilon^2/\varpi^3$. In the similar way as described in the homogeneous case, the Joule-Thomson coefficient of a

trapped gas is given as

$$
\begin{aligned}
u_{JT} &= \frac{V}{6}\left(\frac{\varpi}{T}\right)^3\left[\frac{1}{G_4(z,g)}\right.\\
&\quad\left.-\frac{G_2(z,g)}{4G_4(z,g)G_2(z,g)-3G_3^2(z,g)}\right]\\
&= \frac{1}{T^3}\left[\frac{1}{h_3(\omega_0,g)}\right.\\
&\quad\left.-\frac{2h_1(\omega_0,g)}{8h_3(\omega_0,g)h_1(\omega_0,g)-9h_2^2(\omega_0,g)}\right], \quad (24)
\end{aligned}
$$

with $V=\varpi^{-3}$ [8, 37].

The corresponding analytical expressions for the particle number and isobar heat capacity per particle are

$$N = 2\left(\frac{T}{\varpi}\right)^3 G_3(z,g), \quad (25)$$

$$\frac{C_P}{N} = \frac{4h_3(\omega_0,g)}{3h_2(\omega_0,g)}\left[\frac{8h_3(\omega_0,g)h_1(\omega_0,g)}{3h_2^2(\omega_0,g)}-3\right]. \quad (26)$$

At zero temperature, the particle number is

$$N = \frac{1}{3}\left(\frac{E_F}{\varpi}\right)^3. \quad (27)$$

Substituting equation (27) into equation (25), one gets

$$
\begin{aligned}
6\left(\frac{T}{T_F}\right)^3 G_3(z,g) &= 3\left(\frac{T}{T_F}\right)^3 h_2(\omega_0,g)\\
&= 1. \quad (28)
\end{aligned}
$$

## IV. EFFECTIVE MASS

At very low temperature, the $G_n(z,g)$ can be expanded as [4, 9–11]

$$G_n(z,g) = \frac{(\ln z)^n}{g\Gamma(n+1)}\left[1+\frac{\pi^2}{6}\frac{gn(n-1)}{(\ln z)^2}+\cdots\right]. \quad (29)$$

From equations (20) and (29), the expression of the isochore heat capacity at extremely low temperature can be given by

$$\frac{C_V}{N} = \frac{\pi^2}{2}g^{1/3}\left(\frac{T}{T_F}\right)+\cdots. \quad (30)$$

The zero-temperature effective mass is

$$
\begin{aligned}
\frac{m^*}{m} &= \frac{C_V}{(C_V)_{ideal}}\\
&= g^{1/3}, \quad (31)
\end{aligned}
$$

where $(C_V)_{ideal}=\pi^2 TN/(2T_F)$ is the isochore heat capacity of an ideal Fermi gas at the first-order approximation and $m$ is the mass of non-interacting ideal fermions.

## V. DISCUSSIONS

The numerical results will be given in this section.

The Joule-Thomson coefficient plotted as a function of reduced temperature can be obtained from equations (15) and (23) for a homogeneous gas of anyons. It can also be given by equations (24) and (28) for a trapped anyon gas numerically.

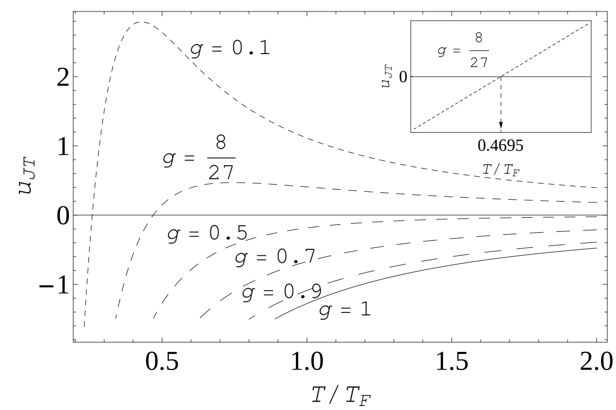


FIG. 1: The Joule-Thomson coefficient $u_{JT}$ is plotted as a function of the reduced temperature for homogeneous gases. The solid curve denotes that for the ideal fermions, and the dashed curves denote the ones for ideal anyons with different statistical parameter $g=0.1, \frac{8}{27}, 0.5, 0.7, 0.9$. Inset shows that the $u_{JT}$ with fixed $g=\frac{8}{27}$ changes its sign at $T/T_F\approx 0.4695$. $m=1$ and $E_F=1$ are chosen for convenience.

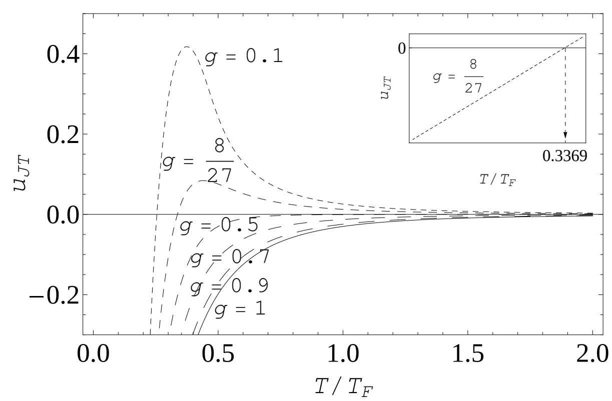


FIG. 2: The line styles are similar to the ones in figure 1 for harmonically trapped gases. The $u_{JT}$ with $g=\frac{8}{27}$ changes its sign at $T/T_F\approx 0.3369$ in the inset. Here, $E_F=1$.

Figures 1 and 2 show that the Joule-Thomson coefficient is larger for smaller values of $g$ at the same temperature for both homogeneous and trapped anyons. Besides, the curves all tend to zero in the high temperature limit as a Boltzmann gas and approach the minus infinity in the strong degenerate limit $T\to 0$. The Joule-Thomson coefficients for both homogeneous and trapped ideal Fermi gases are always negative, which means that there is no Joule-Thomson inversion tempera-

ture for ideal Fermi gas. During the adiabatic expansion, the temperature always increases for an ideal Fermi gas. For ideal anyons, however, there exists a Joule-Thomson inversion temperature, which depends on the parameter $g$ in the fractional exclusion statistics model.

Recently, the strong interaction in two-component ultracold fermions is a hot topic [38–43]. The strongly interacting Fermi gas is called the unitary Fermi gas [40–43]. As a hypothesis, the three-dimensional ideal anyons with fractional exclusion statistics can be used to model the statistical behavior of a real unitary Fermi system [32–36]. This fractional exclusion statistics hypothesis is found to be in good agreement with the experimental results in a harmonic trap [36].

The parameter $g$ in the statistical model can be fixed as $g = \xi^{3/2} = \frac{8}{27}$ [35, 36], where the universal constant $\xi = \frac{4}{9}$ is given by the developed quasi-linear approximation [44–49]. By taking $g = \frac{8}{27}$ as an example, the inversion temperature of a homogeneous unitary Fermi gas is $T/T_F \approx 0.4695$, and it is $T/T_F \approx 0.3369$ for a harmonically trapped unitary system as indicated in figures 1 and 2.

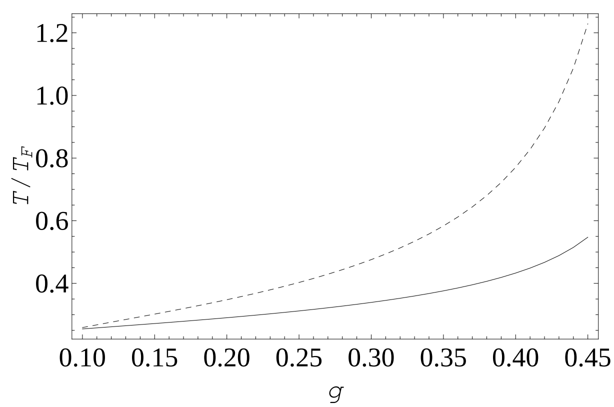


FIG. 3: The reduced Joule-Thomson inversion temperature $T/T_F$ versus $g$. The dashed curve denotes the result of homogeneous anyon gas, and the solid curve represents that of trapped gas.

It is evident from figure 3 that the reduced Joule-Thomson inversion temperature increases with the increasing of the statistical parameter $g$. Further, the reduced inversion temperature of a homogeneous gas is higher than the corresponding one of a trapped gas with the same value of $g$. The difference in the reduced inversion temperatures between a homogeneous gas and a trapped gas becomes larger as $g$ increases.

## VI. SUMMARY

The Joule-Thomson coefficients of a homogeneous and a trapped three-dimensional ideal anyon system have been analyzed within the Haldane fractional exclusion statistics. The results show that the Joule-Thomson coefficient of anyons will overlap with that of the ideal Fermi gas in the high-temperature Boltzmann regime and approach the minus infinity in the strong degenerate limit whatever value of $g$ is chosen. The Joule-Thomson coefficient gets larger for smaller values of the statistical parameter $g$ at the same temperature for both ideal homogeneous and trapped anyons. For ideal fermions, the Joule-Thomson coefficient is negative, which means that there is no Joule-Thomson inversion temperature. However, there exists an inversion temperature for the system obeying Haldane fractional exclusion statistics. The value of the Joule-Thomson inversion temperature depends on the statistical parameter $g$ in the fractional exclusion statistics, and it increases with the increasing $g$. Besides, the reduced inversion temperature of a homogeneous anyon gas is higher than the corresponding trapped one with the same $g$. The deviation between the reduced inversion temperature of a homogeneous gas and that of a trapped gas will become larger and larger as $g$ increases. The effective mass is $m^*/m = g^{1/3}$, which is a function of the statistical factor $g$.

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